 ISSN:2229-6247

Dynamical Behavior in an Innovation Diffusion Marketing Model with Thinker Class of Population

Joydip Dhar[1], Mani Tyagi[2] and Poonam sinha[2]
[1]ABV-Indian Institute of Information Technology and Management
Gwalior-474010(M.P.), India
[2]Department of Applied Sciences, Government Model Science College,
Gwalior-474001(M.P), India

ABSTRACT

The paper is concerned with a product marketing through advertising where a firm devotes a fixed proportion of sales to advertising. Here we proposed a mathematical model with three classes of population, namely, (i) adopter class (ii) potential buyer i.e., non-adopter class and (iii)the thinker class i.e., those buyer who are exposed to advertisement. Moreover, the thinker class population will become adopter after some time lag(i.e. delay).The dynamical behavior of the system is discuss using classical stability theory and oscillatory character of the system is explored using Hopf-bifurcation analysis. Finally a numerical example is given in support of theoretical analysis.

1. INTRODUCTION

Since the publication of the Bass model in [1], research on the modeling of the diffusion of innovations has resulted in a body of literature consisting of several dozen article, books and assorted other publication. Attempts have been made to reexamine the structural and conceptual assumptions and estimation issues underlying the diffusion models of new product acceptance. The authors evaluate these developments for the past two decades. They conclude with a research agenda to make diffusion models theoretically more sound and practically more effective and realistic. The diffusion of an innovation traditionally has been defined as the process by which that innovation "is communicated through certain channels over time among the members of a social system." As a theory of communications, diffusion theory's main focus is on communication channels, which are the means by which information about an innovation is transmitted to or within the social system.

Since its introduction to marketing in the 1960s [1,10-12], innovation diffusion theory has sparked considerable research among consumer behavior , marketing management and management and marketing science scholars. Researchers in consumers behavior have been concerned with evaluation the applicability of hypotheses developed in the general diffusion area to consumer research [7]. The marketing management literatures has focused on the implication of these hypotheses for targeting new product prospects and for developing marketing strategies aimed at potential adopters [8-9] Researchers in management and marketing science have contributed to the developments of diffusion theory by suggesting analytically models for describing and forecasting the diffusion of an innovation in a social system. More recently, this literature also has been concerned developing normative guidelines for how an innovation should be diffused in social system.

Since publication of the Bass model, research on the modeling of the diffusion of innovation in marketing has resulted in an extensive literature. Contribution of this literature through the 1970s were reviewed by [3]. However, in the ensuing decade a plethora of studies has contribution to our understanding of the structural, estimation and conceptual assumptions underlying diffusion models. Modelling of innovation diffusion has been highly applicable to the problems from a wide range of disciplines[2]. Models proposed to deal with this phenomenon at the aggregate and non aggregate levels have evolved over a long period of time. Uncertainty in the models of innovation diffusion may be integrated in the models parameters , its structure , or both . The resulting models may depict the market at the aggregate or disaggregate leavl and be used for predictive or normative purposes. In the present paper we present some modification in the model give by [4]. The paper is organized in 5 sections. Sections 1 deals with the brief introductions and section 2 formulation of the model. The stability analysis is carried out in section 3. Existence of hopf-bifurcation is shown in section 4. Numerical simulation to justify the theoretical result is carried out in section 5.

2. THE MATHEMATICAL MODEL

A three compartment model is considered which consist of adopters class A(t), think class I(t) and non adopters class N(t) population for a particular product marketing. Contact between adopter and non adopter class is through word of mouth. If contact are random the rate of generations of adopter is $\hat{\alpha}$AN, where the terms $\hat{\alpha}$ reflects the word of mouth effectiveness [4]. Moreover, Simon & Sebastian [5], among other provide empirical evidence suggesting that advertising enhances the word-of-mouth effectiveness. Thus, It also assumed that $\hat{\alpha}$= αA, where the constant α reflects the advertising effectiveness. Such a linear relationship makes sense if adopter purchase the product at a constant rate and at a constant price, and the firm devotes a fixed proportion of sales to advertising effectiveness.
There is an external sources of new non-adopters who enter the market at the constant rate K. Adopters are supposed to leave the market at a rate (ξ+η)A where (ξ+η) is the removal constant. Some of them are lost forever at the rate ηA and some of them switch over to other brands, so coming back to non-adopter class at the rate ξA. In the present work we have considered delay in thinker class as the effect of advertisement in non-adopter class is not immediate. There is some time lag τ between non-adopter and adopter. The fraction of thinker class become will adopters after the delay τ is Iγ.

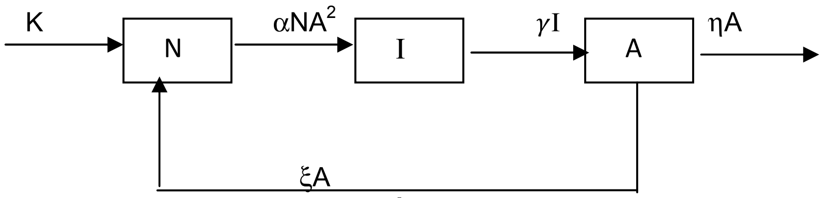

Figure 1

In view of the flow diagram (see Figure 1), the mathematical model is given by following three non-linear ordinary differential equations

$$\frac{dN}{dt}=K-\alpha N A^2+\xi A \quad (1)$$

$$\frac{dI}{dt}=\alpha N A^2-\gamma I(t-\tau) \quad (2)$$

$$\frac{dA}{dt}=\gamma I-\xi A-\eta A \quad (3)$$

3 STABILITY ANALYSIS

The system (1)-(3) has a only one non-trivial positive equilibrium point E (N^*, I^*, A^*) obtained as: $N^*=\frac{\eta(\eta+\xi)}{\alpha K}$, $I^*=\frac{K(\eta+\xi)}{\eta\gamma}$, $A^*=\frac{K}{\eta}$.
Next, we will study the stability of the system (1)-(3). The characteristic equation at E is given by the determinant of the matrix

$$\begin{pmatrix} \lambda+\alpha A^{*2} & 0 & (2\alpha N^* A^*+\xi) \\ \alpha A^* & -(\lambda+\gamma e^{-\lambda\tau}) & 2\alpha N^* A^* \\ 0 & \gamma & -(\lambda+(\eta+\xi)) \end{pmatrix}$$

The characteristic equation becomes

$$\lambda^3+[A_1+\gamma e^{-\lambda\tau}]\lambda^2+[A_2+B_2 e^{-\lambda\tau}]\lambda+[A_3+B_3 e^{-\lambda\tau}]=0 \quad (4)$$

Where
$A_1=\eta+\xi+\alpha A^{*2}$, $A_2=\alpha A^*(\eta+\xi)-2\alpha N^* A^*$, $B_2=\alpha A^{*2}\gamma+\gamma(\eta+\xi)$, $A_3=\xi\gamma\alpha A^{*2}$ and $B_3=\alpha A^{*2}\gamma(\eta+\xi)$.

For local stability of the equilibrium point E, all the roots of (4) must have negative real parts. Analytically it is very difficult to give a condition under which the Eq.(4) will have all the roots with negative real parts. However, when delay is zero(i.e., τ=0) the stability conditions are given by the Routh-Hurwitz criteria as

$(A_1+\gamma)>0$,
$(A_3+B_3)>0$ and
$$(A_1+\gamma)\ (A_2+B_2)-\ (A_3+B_3)>0 \quad (5)$$

Clearly, first two conditions are automatically satisfied and third condition is satisfied under a stringent condition of parameters.

We now discuss the stability of the original delay model about the positive equilibrium, through the hopf-bifurcation analysis in the next section.

4. HOPF-BIFURCATION ANALYSIS

For Hopf- Bifurcation to take place, there must exist a critical time delay τ_{cr} such that $H_1: \lambda_{1,2}(\tau_{cr}) = \pm i\omega(\omega > 0)$ and all other eigen value have negative real part at $\tau = \tau_{cr}$;

$$H_2: Re\left[\frac{d\lambda_{1,2(\tau_{cr})}}{d\tau}\right] \neq 0.$$

For existence of condition H_1, assume that there exists a pair of imaginary roots for (4) i.e. $\lambda = i\omega(\omega > 0)$ is one of its roots.

Substituting in (4) gives

$$A_3\text{-}A_1\omega^2 + [B_3 \cos(\omega\tau) - \gamma\omega^2 \cos(\omega\tau) + B_2 \sin(\omega\tau)] + i[A_2\omega - \omega^2 + B_2\omega \cos(\omega\tau) + \gamma\omega^2 \sin(\omega\tau) - B_3 \sin(\omega\tau)] = 0 \quad (6)$$

Separating the real and imaginary parts, we have

$$(B_3\text{-}\gamma\omega^2) \cos(\omega\tau) + B_2\omega \sin(\omega\tau) = A_1\omega^2 - A_3 \quad (7)$$

$$B_2\omega \cos(\omega\tau) - (B_3 - \gamma\omega^2) \sin(\omega\tau) = \omega^3 - A_2\omega \quad (8)$$

Eliminating $\sin(\omega\tau)$ and $\cos(\omega\tau)$ from the (7)-(8), the following polynomial equation in ω can be obtained.

$$\omega^6 + (A^2 - 2A_2 - \gamma^2)\omega^2 + (A_2^2 + 2B_3\gamma - 2A_1A_2 - B_2^2)\ \omega^2 + (A_3^2 - B_3^2) = 0 \quad (9)$$

Letting $\upsilon = \omega^2$; $Z_1 = A_1^2 - 2A_2^2$; $Z_2 = A_2^2 + 2B_3\gamma - 2A_1A_2 - 2B_2^2$; $Z_3 = A_3^2 - B_3^2$

$$\varphi(\upsilon) \equiv \upsilon^3 + Z_1\upsilon^2 + Z_2\upsilon + Z_3 = 0. \quad (10)$$

Let $\upsilon_{cr} = \omega_{cr}^2$ be a positive roots of (10), then the critical value of delay are given by

$$\cos \omega\tau = \frac{(B_2 - A_1\gamma)\omega^4 + (A_3\gamma - A_2B_2 + A_1B_3)\omega^2 - A_3B_3}{(B_2 - \gamma\omega^2)^2 + B_2^2\omega^2} \text{ (using (7) and (8))}$$

Thus, if we denote

$$\tau_m = \frac{1}{\omega}\left[\cos^{-1}\frac{(B_2 - A_1\gamma)\omega^4 + (A_3\gamma - A_2B_2 + A_1B_3)\omega^2 - A_3B_3}{\gamma^2\omega^4 + (B_2^2 - 2B_2\gamma)\omega^2 + B_2^2} + 2m\pi\right] \quad \text{m=0,1,2...} \quad (11)$$

For Hopf –bifurcation to occur, (10) has at least one positive real root for υ. The next lemma states the conditions under which (10) has none, at least one or more positive roots.

Lemma 1. The polynomial (10) has

(1) no positive roots if $Z_3 \geq 0$ and $\Delta = Z_1^2 - 3Z_2 \leq 0$;
(2) at least one positive root if $Z_3 < 0$;
(3) positive roots iff $Z_3 \geq 0$ and $\Delta = Z_1^2 - 3Z_2 > 0; \varphi(\upsilon_1) \leq 0$,

$\upsilon_1 = \frac{-Z_1 + \sqrt{\Delta}}{3} > 0.$

Proof: This lemma can easily be proved by considering the sign and extremum of $\varphi(\upsilon)$.

Now we will establish the condition: (H_2): $\mathrm{Re}\left[\frac{d\lambda_{cr}}{d\tau}\right] \neq 0$

Taking the derivative λ of with respect to τ in (4), it is easy to obtain:

$$(3\lambda^2 + 2A_1\lambda + A_2)\frac{d\lambda}{d\tau} + (2\lambda\gamma + B_2)e^{-\lambda\tau}\frac{d\lambda}{d\tau} - (\lambda^2\gamma + \lambda B_2 + B_3)\tau e^{-\lambda\tau}\frac{d\lambda}{d\tau}$$
$$-(\lambda^3\gamma + \lambda^2 B_2 + B_3\lambda)e^{-\lambda\tau} = 0,$$

$$\frac{d\lambda}{d\tau}\left[(3\lambda^2 + 2A_1\lambda + A_2) + (2\lambda\gamma + B_2)e^{-\lambda\tau} - (\lambda^2\gamma + \lambda B_2 + B_3)\tau e^{-\lambda\tau}\right]$$
$$= (\lambda^3\gamma + \lambda^2 B_2 + B_3\lambda)e^{-\lambda\tau},$$

$$\frac{d\lambda}{d\tau} = \frac{(\lambda^3\gamma + \lambda^2 B_2 + B_3\lambda)e^{-\lambda\tau}}{(3\lambda^2 + 2A_1\lambda + A_2) + (2\lambda\gamma + B_2)e^{-\lambda\tau} - (\lambda^2\gamma + \lambda B_2 + B_3)\tau\, e^{-\lambda\tau}}$$

Then

$\left(\frac{d\lambda}{d\tau}\right)^{-1}=\frac{(-3\omega^2+2A_1 i\omega+A_2)+(2i\omega\gamma+B_2)(\cos\omega\tau-i\sin\omega\tau)}{(-\omega^3 i\gamma-\omega^2 B_2+i\omega B_3)(\cos\omega\tau-i\sin\omega\tau)}-\frac{\tau}{i\omega}$

$=\frac{(A_2-3\omega^2+2\omega\gamma\sin\omega\tau+B_2\cos\omega\tau)+i(2A_1\omega+2\omega\gamma\cos\omega\tau-B_2\sin\omega\tau)}{(-\omega^3\gamma\sin\omega\tau-\omega^2 B_2\cos\omega\tau+\omega B_3\sin\omega\tau)+i(-\omega^3\gamma\cos\omega\tau+\omega^2 B_2\sin\omega\tau+\omega B_3\cos\omega\tau)}-\frac{\tau}{i\omega}$

Let

Q=$(-\omega^3\gamma\sin\omega\tau-\omega^2 B_2\cos\omega\tau+\omega B_3\sin\omega\tau)^2+(-\omega^3\gamma\cos\omega\tau+\omega^2 B_2\sin\omega\tau+\omega B_3\cos\omega\tau)^2$,

QxRe$\left(\frac{d\lambda}{d\tau}\right)^{-1}$=$[(A_2-3\omega^2+2\omega\gamma\sin\omega\tau+B_2\cos\omega\tau)][(-\omega^3\gamma\sin\omega\tau-\omega^2 B_2\cos\omega\tau+\omega B_3\sin\omega\tau)]$+$[(2A_1\omega+2\omega\gamma\cos\omega\tau-B_2\sin\omega\tau)]$

$\times[(-\omega^3\gamma\cos\omega\tau+\omega^2 B_2\sin\omega\tau+\omega B_3\cos\omega\tau)]$

Noting that

sing$\left\{Re\left(\frac{d\lambda}{d\tau}\right)_{\tau=\tau_0}\right\}$ = sing$\left\{\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_0}\right\}$.

Now, we can employ a result from [6] to analyze (4), which is, for the convenience of the reader.

Theorem 4.1 The positive equilibrium point of system (1)-(3) is absolutely stable if coefficients of (10) satisfy the following conditions:

(1) $Z_3 \geq 0$, $\Delta = Z_1^2-3Z_2 \leq 0$;

Theorem 4.2. The positive equilibrium point of system (1)-(3) is conditionally stable provided the coefficients of (10) satisfy the one of the following conditions:

(1) $Z_3 < 0$;

(2) $Z_3 \geq 0$, $\Delta = Z_1^2-3Z_2 > 0$ and $\psi(\upsilon_1) \leq 0$, for $\upsilon_1 = \frac{-Z_1+\sqrt{\Delta}}{3} > 0$.

Theorem 4.3. Hopf bifurcation exists for the system (1)-(3) if $\psi'(\upsilon_{cr}) \neq 0$ for $\upsilon_{cr} = \omega_{cr}^2$ and one of the following conditions is satisfied:

(1) $Z_3 < 0$;

(2) $Z_3 \geq 0$, $\Delta = Z_1^2-3Z_2 > 0$ and $\psi(\upsilon_1) \leq 0$, for $\upsilon_1 = \frac{-Z_1+\sqrt{\Delta}}{3} > 0$.

5. NUMERICAL SIMULATION

In this section, we present numerical results of system (1)-(3) at different values of τ. First we verify the nonzero equilibrium point is stable and the existence of Hopf-bifurcation in the system by taking the parametric values as: K=0.5; η=0.1089; ξ=0.012; α=0.15; γ=0.13.

In this case, the system has unique equilibrium point (0.1755, 4.2700, 4.5914). The above result has been shown numerically in figures 2-5 In figures 2, Here the critical value of delay parameter is $\tau \approx 10.9412$. We observed that the equilibrium was stable, when $\tau = 10.8$, but when it crossed the threshold value τ =10.9412, the above system showed Hopf-bifurcation, which is shown in figures 3-5 for $\tau = 11.1$,11.5,11.7 respectively.

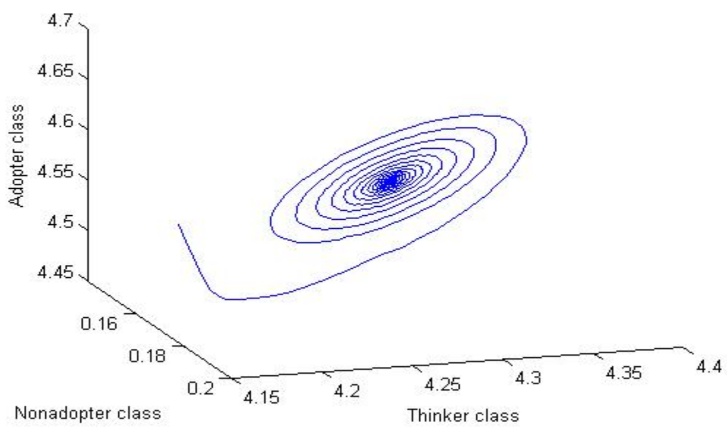

Figure 2: Phase-space diagram for non-adopter, thinker and adopter at τ =10.8

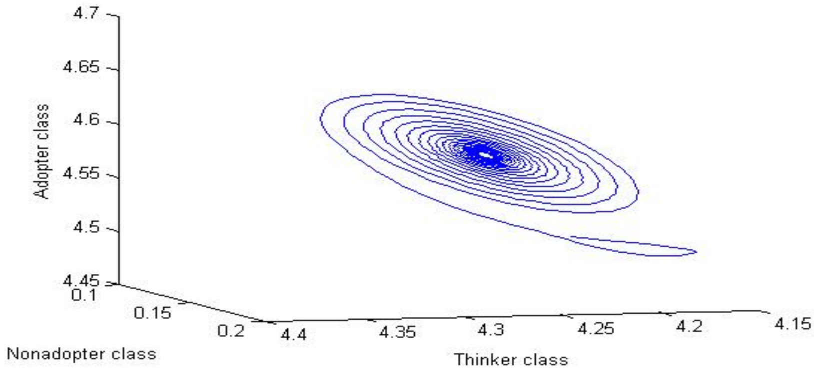

Figure 3:Phase-space diagram for non-adopter, thinker and adopter at $\tau = 11.1$

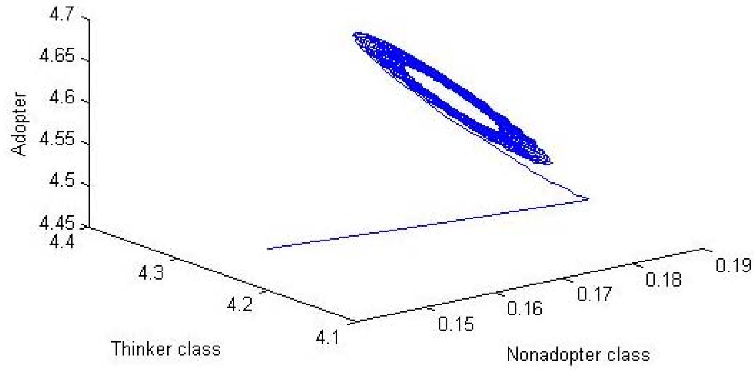

Figure 4: Phase-space diagram for non-adopter, thinker and adopter at $\tau = 11.5$

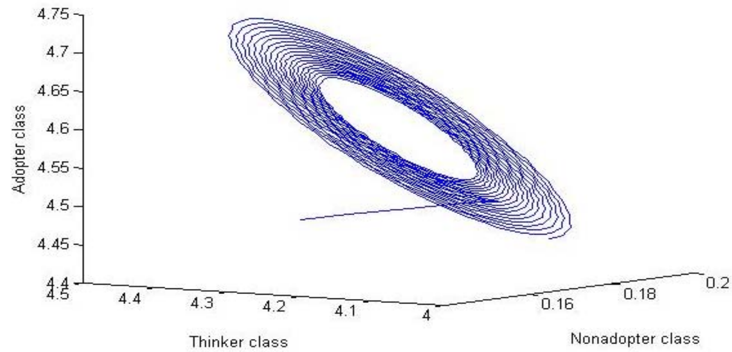

Figure 5: Phase-space diagram for non-adopter, thinker and adopter at $\tau = 11.7$

CONCLUSION

In the present paper a product marketing through advertising where a firm devotes a fixed proportion of sales adverting is considered. A time delay three compartment population model consisting of adopter class, thinker class and non-adopter class has been formulated and analyzed. It is assumed that the thinker class population becomes adopter class after some time delay τ because the effect of advertisement in non-adopter class is not immediate. The resulting system analyzed both analytically and numerically. The stability analyses has been carried out about the unique positive equilibrium point. The system is stable without delay under some condition involving parameters. It is established that the system is stable without delay but by introducing delay parameter, Hopf-bifurcation takes place due to which, the system stable in the range $\tau \epsilon [0, \tau_r)$ and beyond that it unstable and oscillatory character of the system is explored. Using this model one can think new product marketing strategy with respect to advertising intensity.

REFERENCES

[1]. Bass, F. M. [1969] " A New Product Growth Model For Consumer Durables," Management Sci.15 215-227.

[2]. Rogers, Everett M.[1995)] Diffusion of Innovations(4th ed.) New York: Free Press.

[3]. Mahajan Vijay, and Eitan Muller [1979]. "Innovation Diffusion and New-Product Growth Models in Marketing."Journal of Marketing 43(Fall),55-68.

[4]. Feichtinger,G.[1995] "Chaotic Behavior In an Advertising Diffusion Model."International Journal of Bifurcation and Chaos, Vol.5,No.1 225-263.

[5]. Simon, H.& Sebastian, K.H.[1987] "Diffusion and Advertising: The German Telephone Company," Management Sci.33,451-466.

[6]. S.Ruan. J. Wei, [2003] on The Zeros of Transcendental Function With Applications to Stability of Delay Differential Equation With Two Delays, Dyn. Contin Discrete Impuls.Syst.Ser A Math.Anal,10 863-874.

[7]. Gatignon and Thomas S. Robrtson [1985]"A Propositional Inventory for New Diffusion Research"Journal of Consumer Research, 11(march),849-67.

[8]. Engel, James F., Roger D. Blackwell, and paul w. Miniard [1990], Consmer Behavior. Hinsdale Il: Dryden Press

[9]. Kotler P. and Zaltman G.,"Targeting Prospects for a New Product,"Journal of Advertising Research,16(feb.1976),7-18.

[10]. Robertson,A.,[1967] The nature of quantitative genetic variation, pp. 265-280 in Heritage from mendel, edited by A. Brink University of Wisconsin Press, Madison.

[11]. Silk, Alvin J (1966), overlap among Self-Degignated opinion leader : A study of selected Dendal product and service. “Journal of Marketing Research, 3 (Augtst), 255-259

[12]. Arndt, J., 1967. Role of Product – Related Conversion in the Diffussion of a New Products. Journal of Marketing 4 (August), 291-295.